

M-Theory Origin of Mirror Symmetry in Three Dimensional Gauge Theories

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ABSTRACT

We present M-theory compactifications on $K_3 \times K_3$ with membranes near the A_n or D_n singularities of the K_3 spaces. By realizing each of these compactifications in two different ways as type I' models with 2- and 6-branes, we explain the three-dimensional duality between gauge theories recently found by Intriligator and Seiberg. We also find new pairs of dual gauge theories, which we briefly describe.

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1 Introduction

A recent study of three dimensional gauge theories with $N = 4$ supersymmetry, from the viewpoint of both field [1, 2] and string [3] theory, led to the discovery [4] of a new duality between two classes of gauge theories. The first one is a $U(1)$ gauge theory with $n + 1$ electrons, or $SU(2)$ with n quarks. The global symmetry is associated to the ADE classification: $SU(n + 1) = A_n$ and $SO(2n) = D_n$, respectively. The second one is the gauge theory associated with Kronheimer's hyper-Kähler quotient construction of ALE spaces [5]. The duality exchanges theories corresponding to the same ADE group-singularity.

$N=4$ theories are the dimensional reduction of $N=1$ theories in six dimensions. Their global R-symmetry, $SO(4) \cong SU(2)_L \times SU(2)_R$, can be related to the $SU(2)_R$ six-dimensional R-symmetry, and to the $SU(2)_L$ rotation of the three scalars obtained in the dimensional reduction of the vector. Both the Higgs and the Coulomb branch are hyper-Kähler manifolds, and the Higgs branch is not renormalized by quantum effects. Two kinds of coupling constants can be added: masses (vector of $SU(2)_L$, they lift part of the Higgs branch), and Fayet-Iliopoulos terms (vector for $SU(2)_R$, they lift part of the Coulomb branch).

The duality discovered in [4], as usual in non-finite theories, relates long distance effective theories. It simultaneously exchanges the Coulomb branch with the Higgs branch, the mass terms with the Fayet-Iliopoulos terms, and $SU(2)_R$ with $SU(2)_L$.

In this paper we want to relate this picture to a string theory compactification in which this duality is reinterpreted as a string-string duality, and the parameters are identified with the VEVs of spacetime background fields. We will also gain a deeper understanding of this duality by considering a compactification of M-theory on $K_3 \times K_3$, with all membranes near an $A_k \times A_n$ or a $D_n \times D_k$ singularity of $K_3 \times K_3$. In M-theory, the three-dimensional duality follows from realizing in two different ways the same M-theory configuration in terms of type I' models with 2- and 6-branes. Besides its simplicity, this M-theory explanation can be implemented in details in terms of a fairly simple construction involving 6- and 2- D-branes in type IIA theory on $T^3/\Omega Z_2 \times R^4/Z_n$. This must be contrasted with another space-time explanation of the duality, which uses T-duality between type IIA and IIB string theory. In the latter approach, the identification of the Calabi-Yau spaces giving rise to the three-dimensional gauge theories is far from trivial.

Our M-theory explanation of the dualities in ref. [4] can be easily generalized to obtain a larger class of $N = 4$ gauge-theory dual pairs, which are related to the quiver construction in ref. [6].³

³As we were completing this work, a paper [7] appeared which covers related topics. In particular, [7] identifies the Calabi-Yau pairs that explain some of the dualities of ref. [4] via type IIA-IIB T-duality, and construct new examples of mirror pairs starting from quivers diagrams. As we will comment below, some of these new examples

Both theories in [4] can be interpreted as the world-volume theory for a probe [8, 9, 10] in a string spacetime context. A more general construction [6], associated to quiver diagrams, contains both of them as subsectors, and, as we will see, is the natural setting for explaining the duality and for finding new dual pairs. For simplicity, we will begin in section 2 and 3 by reviewing the construction of the theories in [4]. After having understood these building-blocks, it will be easy to construct the general quiver theory in section 4 and to show how to get the result of ref. [4]. In section 5 we will generalize this construction to find new dual pairs.

2 The Type I' Probe

Let us start with what we will call in the following the type I'-probe theory. Following [3], we will consider type IIA on $T^3/\Omega Z_2$, where Z_2 changes the sign to all coordinates in $T^3/\Omega Z_2$. This orientifold is better known as type I' theory, the T-dual of type I. There are 16 D-6-branes (plus 16 images under ΩZ_2) which fill the non-compact space.⁴ Type I' on $T^3/\Omega Z_2$ is conjectured to be dual to M-theory compactified on K_3 . The gauge group is generically $U(1)$ ¹⁶ but it is enhanced to $U(n)$ when the positions of n 6-branes coincide, and to $SO(2n)$ when n 6-branes and their n images under ΩZ_2 coincide at an orientifold point. At the corresponding point in moduli space, M-theory is conjectured to develop an A_{n-1} and a D_n singularity, respectively. Consider now a 2-brane probe transverse to the compact space.⁵ There is generically a $U(1)$ gauge theory on the world-volume, which is enhanced to $SU(2)$ when the probe meets its image at an orientifold point. The three transverse scalars in $T^3/\Omega Z_2$ are the partners of the gauge field, the other four form a hypermultiplet, X , which is decoupled [11, 12]. The open strings which connect 2- and 6-branes are hypermultiplets ϕ_i , $i = 1, \dots, n$, on the world-volume of the probe and have a mass proportional to the distance between the branes. It is now easy to realize a combined spacetime-probe situation in which the probe world-volume describes exactly the three dimensional theory we are looking for [9]. In the region of moduli space near a $U(n)$ enhanced spacetime gauge symmetry, a probe in the region of the n spacetime branes describes a world-volume gauge theory of $U(1)$ with n electrons, while near an $SO(2n)$ spacetime gauge symmetry, the world-volume theory is $SU(2)$ with n quarks. The spacetime gauge symmetry becomes the global flavor symmetry on the world-volume.

Let us consider in some detail the case of $U(1)$ with n electrons. The masses of the electrons are the positions m_i of the n 6-branes in $T^3/\Omega Z_2$, one of them can be eliminated by shifting the origin of the Coulomb branch. When all masses are different, the Higgs branch of the theory is

can be naturally understood in the context of our construction.

⁴Their position in $T^3/\Omega Z_2$ corresponds to the Wilson lines of $SO(32)$ in the original type I picture.

⁵In M-theory this is a membrane transverse to K_3 , while in the T-dual type I picture it is the 5-brane corresponding to a small instanton wrapped around T^3 .

completely lifted. Let us denote with \vec{x} the position of the probe in $T^3/\Omega Z_2$. We will work with a very large $T^3/\Omega Z_2$ and well far away from the orientifold points. The metric in the Coulomb branch is corrected by quantum effects to [1, 4]:

$$ds^2 = g^2(\vec{x})(dt + \vec{\omega} \cdot d\vec{x})^2 + g(\vec{x})^{-2} d\vec{x} \cdot d\vec{x}, \quad (1)$$

with

$$g^{-2}(\vec{x}) = g_{cl}^{-2} + \sum_{i=0}^{n-1} \frac{1}{|\vec{x} - \vec{m}_i|}, \quad \vec{\nabla}(g^{-2}) = \vec{\nabla} \times \vec{\omega}. \quad (2)$$

The classical Coulomb branch is $R^3 \times S^1$ where the radius of S^1 is given by g_{cl} . The metric is 1-loop exact [1], and changes the topology at infinity to R^4/Z_n , thanks to $\vec{\omega}$. The metric in eq. (1) is identified locally with the metric on K_3 in the M-theory description; the 2-brane probe is identified with a membrane, whose world-volume theory indeed describes a sigma model on K_3 with metric given by eq. (1). For generic masses, the 6-branes are separated and this metric (as well as the corresponding K_3) is smooth. When masses coincide, there is an A_{n-1} singularity with non-trivial metric [1, 13]; only when g_{cl} goes to infinity, does this correspond to a flat R^4/Z_n space. Note that the limit $g_{cl} \rightarrow \infty$, for generic \vec{m}_i , gives an ALE space (see eq. (3) below).

The metric in eq. (1) has been obtained with a world-volume computation, but it could have been obtained as well using spacetime considerations. The 2-brane probes the value of the spacetime dilaton at the point \vec{x} [10, 14]. The spacetime 6-branes provide delta function sources for the equation of motion of the dilaton, which now depends on the three coordinates of $T^3/\Omega Z_2$; the solution of the Laplace equation in three dimensions with sources in \vec{m}_i gives to the dilaton a dependence on the inverse of the distance exactly as in eq. (2). The coupling constant g_{cl} corresponds to the asymptotic value of the dilaton.

When the masses coincide (and thus can be put to zero by shifting the origin of \vec{x}), the Coulomb branch has a singularity from which a $n - 1$ -dimensional (in quaternionic unity) Higgs branch starts. This has been interpreted [11, 8] as the moduli space of $U(n)$ instantons on R^4 . In fact, the Coulomb branch describes the phase in which the 2-brane is separated from the background branes. To go to the Higgs branch we need to tune \vec{x} and \vec{m}_i to zero; this implies that the background branes coincide –thereby giving a $U(n)$ spacetime gauge group– and that the probe meets the 6-branes. In this configuration, the probe is located at a point on the (coinciding) background branes, and represents a zero-size instanton of their $U(n)$ world-volume gauge group. The gauge field coupling to the RR background three form A^3 reads $F \wedge F \wedge A^3$. This shows indeed that we can trade a source for A^3 (the 2-brane) for an instanton [8]. We can now give expectation value to the hypermultiplets ϕ_i ; this corresponds to give a finite, nonzero size to the instanton. This mechanism applies in general to p and $p + 4$ parallel branes. The

Higgs branch is not corrected by quantum effects and can be determined classically by solving the algebraic equations for the D-terms.

The D_n case is quite similar. Spacetime branes and probe meet now near an orientifold point. The Coulomb branch describes a space which at infinity has the topology of R^4/D_n , and is smooth for generic masses. When the spacetime branes coincide, there is a singularity from which a one-dimensional Higgs branch starts, corresponding to the moduli space of $SO(2n)$ instantons. In the type I description this is exactly the small instanton described in [11].

3 The Kronheimer Construction

Consider now the gauge theory corresponding to the Kronheimer construction. Using the same notations as in [4], the gauge group is $K_G = (\prod_{i=0}^r U(n_i))/U(1)$, where i runs over the nodes of the extended Dynkin diagram of the group G of rank r , and n_i is the Dynkin index of the node. The diagonal $U(1)$ is not gauged. The hypermultiplet content is $\oplus_{ij} a_{ij}(n_i, n_j)$, where a_{ij} is the adjacency matrix for the extended Dynkin diagram.

If G is simply laced, this gives a construction of the ALE spaces. There is a one-dimensional Higgs branch. The solution of the D-term equations gives the singular space R^4/Γ_G , where Γ_G is the discrete group associated to G . We can blow up the singularity by introducing Fayet-Iliopoulos terms $\vec{\zeta}$ in the gauge theory, one for each $U(1)$, with the constraint that their sum is zero. The solution of the D-term equations determines the metric on the resolved ALE space, which depends on the parameters $\vec{\zeta}$. For instance, in the natural variables which solve the D-term equations, the metric for the Z_n case is given by:

$$ds^2 = g^2(\vec{x})(dt + \vec{\omega} \cdot d\vec{x})^2 + g(\vec{x})^{-2} d\vec{x} \cdot d\vec{x}, \quad (3)$$

with

$$g^{-2}(\vec{x}) = \sum_{i=0}^{n-1} \frac{1}{|\vec{x} - \vec{m}_i|}, \quad \vec{\nabla}(g^{-2}) = \vec{\nabla} \times \vec{\omega}. \quad (4)$$

These $N = 4$ gauge theories can be constructed in terms of D-branes [6, 15, 16]. Consider type IIA “compactified” on the space R^4/Γ_G and put a certain number of 2-brane probes transverse to the singular space. We need $g = |\Gamma_G|$ branes in order to have a faithful representation of the discrete group. If we project out the Chan-Paton factor of the gauge fields and of the hypermultiplets –corresponding to the positions of the probes in R^4/Γ_G – with the g -dimensional regular representation of the orbifold group, we obtain exactly the spectrum of the K_G gauge theory. In the Z_n case, that we shall study next, the regular representation (which is always block diagonal, and contains every representation a number of times equal to its dimension) is $\gamma(\zeta) = \text{diag}\{1, \zeta, \dots, \zeta^{n-1}\}$ where $\zeta = \exp(2\pi i/n)$. The bosonic fields on the world-volume are the gauge fields A_μ , together with their scalar partners (the transverse position \vec{x} in $T^3/\Omega Z_2$),

and the hypermultiplets X , describing positions in the ALE space. We will write them as the two complex scalars, X^1, X^2 , which diagonalize the action of Γ_G . The $n \times n$ Chan-Paton factors of the gauge fields and the two complex scalars must satisfy:

$$\begin{aligned}\gamma \lambda_{A_\mu} \gamma^{-1} &= \lambda_{A_\mu} \\ \gamma \lambda_{X^1} \gamma^{-1} &= \zeta \lambda_{X^1} \\ \gamma \lambda_{X^2} \gamma^{-1} &= \zeta^{-1} \lambda_{X^2},\end{aligned}\tag{5}$$

giving non-zero entries only for A_{ii} , together with its three scalars partners \vec{x}_i , and $X_{i-1,i}^1, X_{n-1,1}^1, X_{i,i-1}^2, X_{1,n-1}^2$. These fields describe an N=4 gauge multiplet $U(1)^n$, and hypermultiplets charged only under adjacent $U(1)$ s (the first and the last $U(1)$ are considered adjacent, in the spirit of the extended Dynkin diagram construction). This is exactly the spectrum of $K_{SU(n)}$.

The singular space can be blown up by turning on VEVs of twisted fields. There are $n - 1$ twisted vector fields $A_\mu^{Bi}, i = 1, \dots, n - 1$, coming from the orbifold point. It can be easily seen from the conformal field theory on the orbifold that these are RR fields.⁶ Since they are RR background fields, they couple to the world-volume gauge fields as [8, 6]:

$$\int d^3x \sum_{i=1}^{n-1} A^{Bi} \wedge \text{tr}(\gamma(\zeta^i)F).\tag{6}$$

The supersymmetrization of this coupling implies that the three spacetime partners of A_μ^{Bi} (responsible for the blow-up of the orbifold) behave as FI terms for the branes [6]. In this way we get a natural correspondence between the spacetime orbifold construction and the Kronheimer gauge theory. The motion of a probe on the resolved space should describe a sigma model on the ALE. The Higgs branch of K_G , in which only one hypermultiplet is left massless, provides such a description.

4 The M-Theory Interpretation

At this point, it is not surprising, maybe, that the two gauge theories [4] are related by duality. Start with type I' on $T^3/\Omega Z_2$, with n near-coincident spacetime branes at a generic point, or $2n$ near-coincident branes at an orientifold point. String duality relates this compactification to M-theory compactified on a K_3 which is locally an ALE space (associated to A_n or D_n) [3]. The probe is now a membrane of M-theory. Compactifying on a circle (in a direction transverse to the brane) we get exactly type IIA on an ALE space with a 2-brane probe. The two gauge theories are mapped into each other by the string duality.

⁶A shortcut is to consider that the RR fields come, in the compactification on the resolved space, by reducing the RR 3-form of type IIA on the $n - 1$ spheres that resolve the singularity.

A deeper description, more detailed and symmetric, is obtained by working always in type I' theory. This description will allow us to find the M-theory origin of the three dimensional mirror-symmetry. To this purpose, we must analyze a more general construction.

Consider the theory of the probe in type I' on $T^3/\Omega Z_2$ and “compactify” the remaining four transverse directions on an ALE space. This model has been considered in [6] and, after a T-duality, corresponds exactly to the construction in [12], where now the singular K_3 is replaced by a singular ALE. The non-compactness of the space still allows us to consider the 2-brane as a probe; this implies, for example, that we do not have to impose the tadpole conditions (the 2-brane flux can escape to infinity in the ALE), so that the number of probes and their world-volume gauge group are essentially unconstrained. Using the same notation as in the two examples above, which can be thought of as two limits of the present construction, we now need $2g$ branes to account for the images under ΩZ_2 and Γ_G . The hypermultiplets X , which were decoupled in the type I' example, now become a coupled set of hypermultiplets, describing the position of the system on the ALE space, as in the K_G example. We still have hypermultiplets, ϕ , which correspond to open string connecting 2- and 6-branes. This set of fields now couples in a complicated way. To work out the field content of the world-volume gauge theory, we must first find out the way in which ΩZ_2 and Γ_G act on the Chan-Paton factors. This action is constrained by some consistency conditions listed in [12]. The solution of these conditions and the corresponding world-volume theory can be found in [6], for several particular cases where it is related to a quiver diagram construction.

Consider, to begin with, the case of k near-coincident 6-branes at a generic point on $T^3/\Omega Z_2$, and a compactification on an ALE with an A_n singularity. This means that there is a $U(k)$ spacetime gauge symmetry, and that we do not need to project the probe theory with ΩZ_2 . The Higgs branch of this theory has been interpreted [6] as the moduli space of a $U(k)$ instanton on the ALE space. This follows from the general construction we sketched in the discussion of the type I' example. In [6], it was shown that the solution of the D-term equations indeed corresponds to a mathematical construction of that moduli space, introduced by Kronheimer and Nakajima [17].

Let us work out the field content for the probe world-volume theory. Consider n 6-branes and a compactification on the ALE space corresponding to the discrete group Z_n . ΩZ_2 relates branes which are far away and that contribute only extremely massive fields; since it relates 2-6 open strings with 6-2 strings it does not affect the hypermultiplets ϕ . We only have to project out $n \times n$ Chan-Paton factors by Z_n , whose action as an $n \times n$ unitary matrix has been given in the discussion of the K_G theory. The projection for the gauge fields and the hypermultiplets X is as in eq. (5) and gives the same result: $U(1)^n$ with hypermultiplets charged under adjacent $U(1)$ s. The hypermultiplets $\phi_{i,A}$, which have an index i in the fundamental of the global spacetime

symmetry $U(k)$, and a Chan-Paton index A , are projected by $\phi_{iA} = \gamma(\zeta)_{AB}\phi_{iB}$, giving k extra hypermultiplets charged only with respect to the first $U(1)$.

The VEVs of spacetime fields become world-volume coupling constants. The classical coupling g_{cl} is the asymptotic value of the dilaton; the masses for the ϕ_i are the positions of the spacetime 6-branes (there are only $k - 1$ independent mass vectors since we have the freedom to shift the origin of the Coulomb branch); the FI terms for the X are the $n - 1$ twisted-sector blow-up vector moduli of the ALE. The global R-symmetry group of the N=4 theory is easily identified, by relating it to a dimensional reduction from 10 dimensions. Namely, $SU(2)_L$ corresponds to the rotations on $T^3/\Omega Z_2$, and $SU(2)_R$ corresponds to the subgroup of $SO(4)$ (rotation of the coordinates corresponding to X) unbroken by the ALE. It is clear from this spacetime interpretation that masses are vector of $SU(2)_L$ and FI are vectors of $SU(2)_R$.

Let us write the D-term equations for the Higgs phase of this theory. Since we have implicitly used a trivial action of Z_n on the background branes (which are “wrapped” on the ALE space) we are assuming, according to [6], that we are dealing with the moduli space of $U(n)$ instantons on the ALE that do not break the gauge group at infinity. The D-term equations read (ϕ is decomposed in two complex scalars ϕ^1, ϕ^2):

$$\begin{aligned}
\sum_{j=0}^{k-1} (|\phi_j^1|^2 - |\phi_j^2|^2) + |X_{0,1}^1|^2 - |X_{n-1,0}^1|^2 + |X_{0,n-1}^2|^2 - |X_{1,0}^2|^2 &= \zeta_1^D \\
|X_{i,i+1}^1|^2 - |X_{i-1,i}^1|^2 + |X_{i,i-1}^2|^2 - |X_{i+1,i}^2|^2 &= \zeta_i^D, \quad i = 1, \dots, n-1 \\
\sum_{j=0}^{k-1} (\phi_j^1 \phi_j^2) + X_{0,1}^1 X_{1,0}^2 - X_{n-1,0}^1 X_{0,n-1}^2 &= \zeta_1^F \\
X_{i,i+1}^1 X_{i+1,i}^2 - X_{i-1,i}^1 X_{i,i-1}^2 &= \zeta_i^F, \quad i = 1, \dots, n-1,
\end{aligned} \tag{7}$$

where $n \equiv 0$. The $n - 1$ independent FI terms are associated to a set of collapsing two-spheres, whose intersection matrix is the Cartan matrix of G . This means that in this basis $\sum_i \zeta_i = 0$. By adding up all these equations, it is immediately seen that the contribution of the ϕ and that of the X factorize; in particular the equation for ϕ is not affected by the FI terms, when they satisfies the previous condition. Since we have to mod out by the gauge group, this moduli space is a fibration over the moduli space of a $U(k)$ instanton on R^4 .

The equations determining the vacuum manifold, in a general phase (Higgs or Coulomb) are eq. (7) and:

$$\begin{aligned}
(\vec{x}_0 - \vec{m}_j) \cdot \vec{\sigma}_{AB} \phi_j^B &= 0, \quad j = 0, \dots, k-1, \quad A, B = 1, 2, \\
(\vec{x}_i - \vec{x}_{i+1}) \cdot \vec{\sigma}_{AB} X_{i,i+1}^B &= 0, \quad i = 0, \dots, n-1.
\end{aligned} \tag{8}$$

Using string duality, we can relate this compactification (in a suitable region of moduli space) to M-theory on the product of two large K_3 spaces, that, when the probes are near a singularity,

can be approximated by two ALE spaces. The $k - 1$ masses for the 6-branes parametrize the moduli of the first ALE, while the $n - 1$ FI parameterize the moduli of the second. This M-theory compactification can be described in two different ways in type I' superstrings; namely, as a compactification on $T^3/\Omega Z_2 \times R^4/Z_k$ with n 6-branes or as type I' on $T^3/\Omega Z_2 \times R^4/Z_n$ with k 6-branes. These two type I' models give rise to two different three-dimensional world-volume theories on the probe, which, because of this construction, are obviously dual. To stay in the region of moduli space in which this approximation is valid, we must consider a very large $T^3/\Omega Z_2$ and we must set $g_{cl} = \infty$ (see remarks after eq. (1)), thereby probing the long distance of the world-volume theory. As it should be clear from the spacetime interpretation of the various world-volume fields, the Coulomb branch describes the motion of the probe on $T^3/\Omega Z_2$, that is, on the first ALE, while the Higgs branch is associated with the motion on the second. As we saw before, masses and FI are naturally associated with the first and second ALE, respectively. In the coordinates used above in eq. (3), $SU(2)$ acts on the ALE space by rotating \vec{x} . The $SU(2)_L$ on the probe world-volume is associated naturally with the $SU(2)$ on the first ALE, while the $SU(2)_R$ is associated with the $SU(2)$ on the second. Therefore, this duality exchanges the Coulomb branch with the Higgs branch, $SU(2)_L$ with $SU(2)_R$, and the masses with the FI terms.

Let us perform some simple checks. Turn on both masses and FI terms. This is a completely geometrical compactification of M-theory; thus, we are free to exchange the order of compactification. In type I', the fields ϕ are massive (lifting the Higgs branch for the ϕ) and the FI terms lift all the $U(1)$ s except the diagonal one, under which the ϕ are charged. We are left with the product of the Higgs branch for the theory $K_{SU(n)}$ (depending on the $n - 1$ parameters $\vec{\zeta}_i$) and the Coulomb branch for $U(1)$ with k electrons (depending on the $k - 1$ parameters \vec{m}_i). In the limit in which g_{cl} goes to infinity, this theory, and the one in which k and n (as well as \vec{m}_i and $\vec{\zeta}_i$) are interchanged, correspond to the same M-theory construction. This implies that the Higgs branch for the theory $K_{SU(n)}$ must coincide with the Coulomb branch for $U(1)$ with n electrons, since the duality exchanges masses with FI. This is indeed the simplest example to check, by simple inspection of eqs. (1,3). The check was done in [4]. For large coupling constant both manifolds are ALE spaces. This is indeed what we expect from the M-theory picture: the theory on the membrane is a sigma model on the product of two ALE spaces.

The two theories in [4] are contained as subsectors. They can be easily decoupled by going to infinity on one of the two ALE (where the metric is flat). Going to infinity in the first ALE means sending to infinity the center of mass of the system of branes on $T^3/\Omega Z_2$, $\vec{x}^{cm} = \vec{M}$, while keeping finite or zero the relative positions \vec{x}'_i ($\vec{x} = \vec{M} + \vec{x}'$). This decouples completely the fields ϕ (their mass is proportional to $\vec{x}_1 = \vec{M} + \dots$), but not the fields X (their mass is proportional to $\vec{x}_i - \vec{x}_j$). As we may expect, since the probe is far from the background brane, we are left

with the K_G theory plus a decoupled $U(1)$. With FI terms, the theory is automatically in the Higgs branch (describing the motion on the second ALE). By turning off the FI terms, we can go in the Coulomb branch, which describes, in our picture, a probe that lies at the singularity of the second ALE and moves on the first ALE (now locally flat). Going to infinity on the second ALE means sending to infinity X . We expect on general ground that the D-terms will factorize in a flat metric for X times the Higgs branch for the type I' probe theory. In fact, in the region we are considering, the ALE space is now flat and the instanton moduli space must be the same as the instanton moduli space on R^4 . We are left with the type I' probe theory times a flat sigma model. The Coulomb branch corresponds to the motion of the brane on the first ALE. When the masses are zero, the theory can be in the Higgs branch, which means that the probe lies at the singularity on the first ALE, and moves on the second ALE (now locally flat). This picture is clearly symmetric under the exchange of the two equivalent realizations of the M-theory compactification in terms of type I' models. In other words, the M-theory construction makes three dimensional gauge theory duality manifest.

The point where the Coulomb and Higgs branches of these theories meet, which, as argued in [4], gives a non-trivial interacting three dimensional theory, corresponds to a point in M-theory where the ALE space is singular, and where the probe sits on the singularity. Solitons corresponding to membranes wrapped around vanishing two-spheres are becoming massless there, and should be incorporated in the low-energy description.

It would be interesting to understand the spacetime meaning of the remaining coupling constants: a mass for the Coulomb branch of K_G and a FI term for Higgs branch of the type I' probe theory. As noted in [6], $\sum_i \vec{\zeta}_i$ (which we put to zero before) is a natural candidate for the missing FI term. It would correspond to turning on the VEVs of the partners of the type IIA 1-form. This form is projected out by the type I' projection, but if we are far from the orientifold point the theory is practically type IIA and this field is almost massless (this is consistent with the fact that this FI term can exist only in the A_n serie, where the probe is far from the orientifolds). The D-term equations still factorize but now this FI enters the equations for ϕ , deforming and smoothing the $U(n)$ instanton moduli space, otherwise singular. The corresponding mass term should be related to the overall mass we always put to zero by shifting the origin in the Coulomb branch; the precise implementation of this idea is not yet clear to us.

The D_n series can be discussed along similar lines. Background branes and probes now lives near an orientifold point. The probe theory is $SU(2)$ with n quarks and the K_G is $U(1)^4 \times U(2)^{n-3}$ with hypermultiplets naturally associated with the extended Dynkin diagram. The two dual $N=4$ gauge theories can be obtained in symmetric limits of the M-theory compactification. Instead of giving the details, we will briefly explain in the next section how our construction

gives rise to more general examples of three-dimensional dualities.

5 New Dual Pairs

The previous construction can be generalized in the following ways. First of all, we can compactify on a generic pair of ALE spaces; by exchanging the two ALEs we still find a pair of dual gauge theories. Secondly, we considered complicated gauge theories which reduced, in particular limits, to the pair of dual theories in [4]. Had we not decoupled them, we would have found more complex examples of dual pairs. Thirdly, we can consider several coincident probes. This gives $U(k)$ and $Sp(k)$ groups, corresponding, in some phases, to the k -instanton sector of the spacetime gauge group on some ALE. All these models are related to the quiver diagrams of ref. [6].

Another generalization may be to relax the condition that Γ_G does not act on the background branes. Since the most general quiver construction does not have this restriction, it would be interesting to understand its meaning in terms of our M-theory description.⁷

The generality of this approach should be clear now. Consider a general “compactification” of M-theory on two ALE spaces. The interchange of the two ALEs relates two different gauge theories associated to different quiver diagrams in type I’ theory. To go to the region of moduli space in which the M-theory description is valid, we must set $g_{cl} = \infty$, thereby probing the long distance of the world-volume theory. As we stressed before, the Coulomb branch describes the motion of the probe on the first ALE, while the Higgs branch is associated with the motion on the second. Masses and FI are naturally associated with the first and second ALE, respectively; $SU(2)_L$ is associated naturally with the first ALE, while $SU(2)_R$ is associated with the second. Therefore, this duality exchanges the Coulomb branch with the Higgs branch, $SU(2)_L$ with $SU(2)_R$, and the masses with the FI.

For instance, we can see how some of the models in [7] fit in our picture. They are the natural generalization of the probe- K_G theories to the case of a set of k probes. The dual gauge theories are called A and B, and described as follows. A model: $U(k)$ gauge group with n hypermultiplets in the fundamental and one in the adjoint. B model: $U(k)^n$ gauge group with hypermultiplets in the fundamental of pairs of adjacent factors, and one hypermultiplet in the fundamental of one of the $U(k)$. The model B can be realized in type I’ on the Z_n ALE with a background 6-brane. This corresponds to M-theory on the ALE space times a smooth K_3 , which is locally R^4 , (the metric of the Coulomb branch in type I’, which describes a smooth Taub-Nut space, is flat when $g_{cl} = \infty$). Interchanging the ALEs in the M-theory picture, we get

⁷ The example called $U(k)^n$ in [7] is indeed associated with a pair of quiver diagrams of this more general form. Remarkably enough, a naive application of our method reproduces the dual pairs explicitly checked in [7]. One can speculate that each quiver diagram can be related to a dual one with a generalization of the method.

k probes near n spacetime branes, which give exactly the content of the A model (the adjoint hypermultiplet is simply the position of the probes in R^4). Notice that in trying to recover the B model from M-theory directly, there exists an ambiguity, since superficially we could have put either one or no spacetime branes in the B model. This ambiguity is resolved by mapping M-theory on $R^4/Z_n \times R^4$ into the type I' construction described above. The $Sp(k)$ case in [7] simply corresponds to the D_n serie.

More general dual pairs can be obtained by using two non-trivial ALE spaces, while keeping the branes at finite distance from the centers of *both* ALEs. We simply quote the simplest examples.

Our main example, $R^4/Z_k \times R^4/Z_n$ (here generalized to the case of p coincident probes) leads to a dual pair, in which the first model has a $U(p)^n$ gauge group with hypermultiplets in the fundamental of pairs of adjacent factors, and k hypermultiplets in the fundamental of a $U(p)$, while the second one has a $U(p)^k$ gauge group with hypermultiplets in the fundamental of pairs of adjacent factors, and n hypermultiplets in the fundamental of a $U(p)$. The dimension of the Higgs and Coulomb branches and the number of parameters are summarized in Table 2.

$R^4/D_n \times R^4/Z_k$ is complicated because the implementation of the orientifold projection ΩZ_2 is full of subtleties. The first model consists of type I' on R^4/D_n with k background branes; we need not worry about ΩZ_2 , since the 6-branes are far from the orientifolds, and we easily obtain a theory $U(1)^4 \times U(2)^{n-3}$ with hypermultiplets associated with the extended Dynkin diagram and k hypermultiplets charged under a $U(1)$. The dimension of the Coulomb and Higgs branches are: $d_C = 2n - 2$, $d_H = k$. We can put $n + 1$ FI terms (one is associated with an untwisted modulus $\sum \vec{\zeta}$) and $k - 1$ masses. The dual theory is type I' with $2n$ spacetime branes near the orientifold (n 6-branes plus n images), on R^4/Z_k . The consistency condition that ΩZ_2 and Z_k commute leads to $\gamma_\zeta \gamma_{\Omega Z_2} \gamma_\zeta^T = \chi_\zeta \gamma_{\Omega Z_2}$, where χ_ζ is a phase, equal to 1 for k odd, and to 1 or ζ (a k -th root of unity) if k is even. The world-volume theories are listed in full generality in [6]. Remarkably, the dimension of the Coulomb and Higgs branches of all these theories is compatible with duality, namely: $d_C = k$, $d_H = 2n - 2$. Let us consider now for simplicity the case $k = 2$: in [15] a new consistency condition for the projections was found, implying that the only consistent orientifold theory has $\chi = -1$. The theory with $\chi = -1$ is $U(2)$ with 2 hypermultiplets charged under $U(1)$ and n quarks in the fundamental of $U(2)$. We propose that this theory should be dual to $U(1)^4 \times U(2)^{n-3}$ with hypermultiplets associated with the extended Dynkin diagram and 2 hypermultiplets charged under a $U(1)$. The number of parameters perfectly matches as can be seen in Table 2. The simplest check of this proposed duality is to turn on all the parameters associated to the resolution of the two ALE spaces: from the point of view of M-theory we expect that the world-volume theory is a supersymmetric sigma model on the product of the two ALEs; this can be indeed checked explicitly in both theories.

For arbitrary k we encounter a subtlety: as noted in [15], the ΩZ_2 operator which is naturally associated with the orientifold construction [18] is not the limit of the type I' ΩZ_2 operator of the smooth theory. The orientifold projection, indeed, interchanges sectors twisted by ζ^i with sectors twisted by ζ^{k-i} , projecting out some of the twisted sectors hypermultiplets; this means that only some blow-up parameters can be turned on, and that the orbifold singularity cannot be completely smoothed out. This is consistent with the fact that in [6] it is shown that the D-term equations are consistent only for $\zeta_{n-i} = \zeta_i$. The consistency condition in [15] constrains our choice of χ only for k even. To sum up, the explicit form of the world-volume theory of n 6-branes near an orientifold of T^3/Z_2 , on R^4/Z_k , can be found in [6] and it is summarized in Table 1. Let us simply note that the spacetime blow-up parameters are in one-to-one correspondence with the FI terms allowed in the gauge theory. The generic blow-up of the Z_k models always leaves $[(k-1)/2]$ A_1 singularities; interchanging the compactification order, this implies that in the dual picture the spacetime branes must coincide pairwise, and that the world-volume theory has extra, unbroken global $U(2)$ symmetries. We conjecture, therefore, that the dual gauge theory is still $U(1)^4 \times U(2)^{n-3}$ with hypermultiplets associated with the extended Dynkin diagrams and k hypermultiplets charged under a $U(1)$, but with only at most the $[k/2]$ independent masses that respect the global $U(2)$ symmetries. An alternative possibility is that the dual model is defined by a D_n projection acting also on the spacetime branes. The only change in the world-volume theory is that the k hypermultiplets are now linked to different nodes of the extended Dynkin diagram, and some of them may belong to doublets of some $SU(2)$ gauge groups. This reduces the number of allowed masses again to $[k/2]$ [19]. Whichever the explanation, to obtain the case when masses have generic values one would have to change in a rather mysterious way –to us, at least– the definition of ΩZ_2 .

The results of our analysis are summarized in Tables 1 and 2.⁸ Table 1 gives the field content (gauge group and matter representations, underlined, with multiplicity) for each of the theories considered in this paper, while Table 2 gives the dimensions of their Higgs and Coulomb branches, as well as the number of independent masses and FI terms.

⁸ Note that sometimes there is an additional parameter on the world-volume gauge theory, besides the spacetime blow-up parameters. This extra parameter may be associated with the untwisted mode $\sum \zeta$ on the ALE, as we discussed in section 4, in which case it can be turned on only when we are far from the orientifold point.

Table 1: Field Content of Dual Pairs

Singularity Type	Gauge Group	Matter	Remarks
$Z_k \times Z_n$	$U(p)^n$	$n(\underline{p}^+, \underline{p}^-) + k\underline{p}^+$	p 2-brane probes, \pm are $U(1)$ indices
$Z_k \times D_n$	$U(1)^4 \times U(2)^{n-3}$	$4(\underline{1}^+, \underline{2}^-) + (n-4)(\underline{2}^+, \underline{2}^-) + k\underline{1}^+$	one probe
$D_n \times Z_2$	$U(2)$	$2\underline{1}^+ + n\underline{2}^+$	
$D_n \times Z_k$	$SU(2) \times U(2)^{(k-1)/2}$	$(k-1)/2(\underline{2}^+, \underline{2}^-) + 1\underline{1}^+ + n\underline{2}^+$	k odd
	$U(2)^{k/2}$	$2\underline{1}^+ + (k/2 - 1)(\underline{2}^+, \underline{2}^-) + n\underline{2}^+$	k even

Table 2: Moduli Space and Deformation Parameters of Dual Pairs

Singularity Type	d_H	d_C	\vec{m}	$\vec{\zeta}$	Remarks
$Z_k \times Z_n$	kp	np	k	n	
$Z_2 \times D_n$	2	$2n-2$	1	$n+1$	
$D_n \times Z_2$	$2n-2$	2	$n+1$	1	
$Z_k \times D_n$	k	$2n-2$	$[k/2]$	$n+1$	the k 6-branes are pairwise coincident
$D_n \times Z_k$	$2n-2$	k	$n+1$	$[k/2]$	

To summarize, in this paper we presented an M-theory construction which associates pairs of dual three-dimensional gauge theories to quiver diagrams, and we carried out a preliminary study of some examples. The case $D_n \times D_k$ can also be studied with the techniques presented in this paper, and is presently under investigation [19]. Finally, even though in this paper we considered $R^4/Z_n \times R^4/Z_k$ constructions in which the spacetime branes are not projected under Γ_G , results in [7] seems to indicate that even this restriction could be relaxed [19].

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